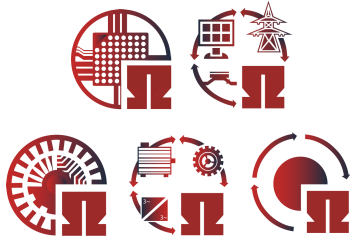


ELSYS Note



Weekly curl

This note introduces the basic idea of two-dimensional electromagnetic finite element analysis by means of a simple U-core with airgap. Starting from Ampère’s law, the formulation is transformed into Maxwell’s differential equation and then rewritten in terms of the magnetic vector potential. The resulting airgap plots of A_z and B_n illustrate how the curl operator links scalar potential and magnetic flux density.

Introduction

Electromagnetic finite element analysis (FEA) is often introduced through governing differential equations, although the physical meaning is usually easier to understand from the integral laws. In this note, a simple U-core with a bar and a small airgap is used to illustrate the main steps from Ampère’s law to a two-dimensional finite element formulation. The geometry is shown in Fig. 1. A current-carrying coil is placed on the middle leg of the U-core, and the magnetic circuit is closed by the bar below the core. This example is particularly useful in teaching because it connects field theory, vector calculus, and finite element implementation in a compact and intuitive way. The airgap is of special interest, since the field is nearly uniform there and can therefore be interpreted directly from the computed results.

Ampère to Maxwell

We start with Ampère’s law in integral form,

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{a},$$

which states that the line integral of the magnetic field intensity along a

closed path equals the total current enclosed by that path. The surface integral is usually easy to evaluate, since it corresponds directly to the enclosed current. The contour integral is less intuitive for students, because the same enclosed current is obtained for many different paths. Hence, if the path length increases, the average magnetic field intensity must decrease.

For finite element analysis, however, the integral form is not sufficient. By applying Stokes’ theorem, the contour integral can be transformed into a surface integral of the curl. Since this relation must hold for arbitrary surfaces, one directly obtains Maxwell’s equation

$$\int_S (\nabla \times \vec{H}) \cdot d\vec{a} = \int_S \vec{J} \cdot d\vec{a},$$

$$\nabla \times \vec{H} = \vec{J}.$$

Material law

In magnetic materials, the flux density and the field intensity are related by

$$\vec{B} = \mu \vec{H}.$$

Hence,

$$\vec{H} = \nu \vec{B}, \text{ where } \nu = \frac{1}{\mu},$$

where ν is the reluctivity. The use of reluctivity is convenient in

numerical formulations, since multiplication is preferred over division. Substituting this relation into Maxwell’s equation gives

$$\nabla \times (\nu \vec{B}) = \vec{J}.$$

The curl

A further simplification is obtained by introducing the magnetic vector potential \vec{A} . In this step, a vector quantity is defined such that its curl gives the magnetic flux density,

$$\vec{B} = \nabla \times \vec{A}.$$

This leads to

$$\nabla \times (\nu \nabla \times \vec{A}) = \vec{J},$$

which is the basis for the magnetostatic finite element formulation.

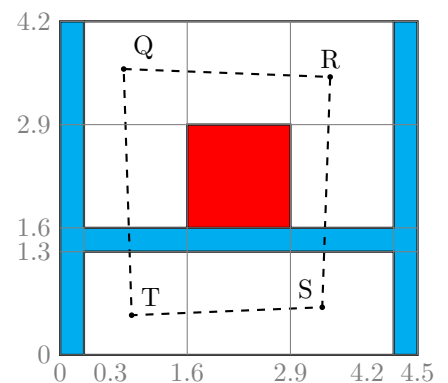


Fig. 1: U-core (values in m)

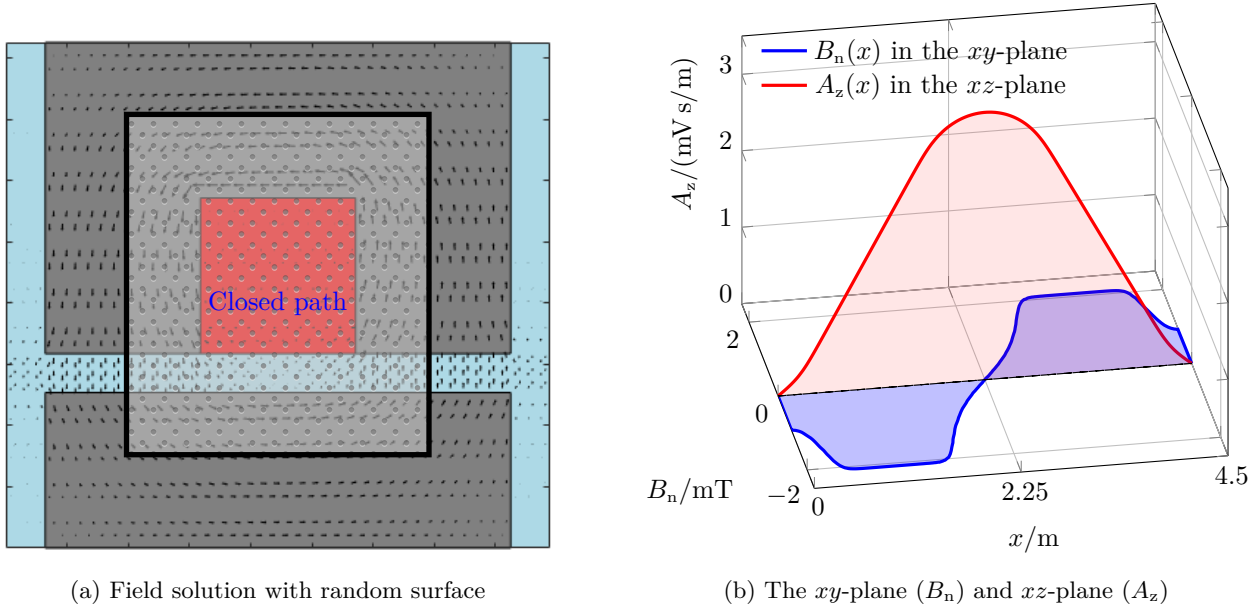


Fig. 2: U-core with bar

2D formulation

For a two-dimensional problem, only the z -component of the magnetic vector potential is required,

$$\vec{A} = \begin{bmatrix} 0 \\ 0 \\ A_z \end{bmatrix}.$$

The magnetic flux density then follows from the curl of \vec{A} . Using the determinant form of the curl, one obtains

$$\vec{B} = \nabla \times \vec{A} = \begin{bmatrix} \frac{\partial A_z}{\partial y} \\ -\frac{\partial A_z}{\partial x} \\ 0 \end{bmatrix}.$$

Thus, the magnetic field components are obtained directly from the spatial derivatives of the scalar quantity A_z . This is one of the key reasons why the vector potential formulation is so useful in electromagnetic FEM.

For most planar magnetostatic problems, first-order triangular elements are sufficient. In this case, each element contributes a 3×3 stiffness matrix whose entries depend only on the nodal coordinates,

the triangle area, and the element reluctivity. After assembly, the global system can be written as

$$[S][A] = [J].$$

The global stiffness matrix is square and symmetric, but singular unless boundary conditions are applied. In practice, Dirichlet boundary conditions are often used by prescribing $A_z = 0$ on the outer boundary.

The airgap results

The computed results along a line in the airgap are shown in Fig. 2. The flux density as a 2D vector is shown in Fig. 2(a). The normal flux-density component B_n is plotted in the xy -plane, whereas the scalar potential A_z is plotted in the xz -plane. This representation emphasizes that the two quantities belong to different geometric surfaces and are connected by the curl operator, as shown in Fig. 2(b).

In the airgap, the flux density is approximately constant. Consequently, the slope of the $A_z(x)$ -curve is nearly constant as well. Since

$$B_y = -\frac{\partial A_z}{\partial x},$$

the gradient of the scalar potential directly determines the magnetic flux density. Near the window of the core and close to the opposite leg, the slope changes, reflecting the local variation of the magnetic field. In this way, the U-core example provides a compact and intuitive illustration of how the magnetic vector potential leads to the field distribution used in finite element analysis.

Right-hand rule

The surface integral over the arbitrary surface in Fig. 2(a) equals 1000 A in this example, i.e.

$$\int_S \vec{J} \cdot d\vec{a} = 1000 \text{ A}.$$

From Fig. 2(b) it can be seen that $A_z > 0$. If the right-hand rule is applied consistently throughout the 2D formulation, the scalar potential has the same direction as the current density. Adhering to electromagnetic sign conventions therefore simplifies both the formulation and the interpretation of the results.