ELSYS Note

The role of the flux per pole

This note explores the role of the flux per pole in electrical machines. By examining the spatial and temporal behavior of the air-gap flux density, and applying Faraday's law, the connection between flux linkage and induced voltage is established – highlighting how machine design impacts terminal voltage and performance.

Design Overview

When designing electrical machines, one of the primary considerations is the terminal voltage, often expressed in its simplest form as

$$U_{\mathrm{rms}} = \omega \cdot \psi_{\mathrm{rms}}$$

This compact equation relates the induced voltage U to the electrical angular velocity ω and the flux linkage ψ. While this relationship appears straightforward, understanding the full theory behind it requires a deep integration of electromagnetics, geometry, and dynamic system behavior.

Rotating magnetic field

In three-phase electrical machines, the first foundational step is to create a rotating magnetic field in the air-gap. This is achieved by arranging three stator windings mechanically spaced 120° apart around the air-gap periphery. To generate a rotating flux, these windings must also be supplied with three-phase currents that are electrically phase-shifted by 120° from one another. The spatial and temporal phase shifts together result in a rotating magnetic field – a sinusoidal wave that travels uniformly around the stator's inner circumference.
This rotating field is the fundamental mechanism behind torque generation and voltage induction in synchronous and induction machines.

Flux density wave

The resulting air-gap flux density in a three-phase machine takes the form of a traveling wave, given by the expression:

$$B(x,t) = \hat{B} \cdot \cos\left(\frac{\pi}{\tau_{\mathrm{p}}} \cdot x - \omega t\right)$$

This wave represents a rotating sinusoidal distribution of the magnetic flux density around the air-gap. For a given magnetomotive force (mmf), the amplitude $\hat{B}$ remains constant, indicating that the flux density wave maintains a fixed shape and amplitude as it rotates at angular velocity ω.
Fig. 1 illustrates this rotating field at three different time instants, showing how the peak of the flux wave travels around the air-gap – a key feature in electromagnetic torque production and voltage generation.
Due to the spatial symmetry of the flux density wave, it is sufficient to analyze the machine behavior over a single wavelength, or more specifically, over an interval of twice the pole pitch, i.e. $x \in [-\tau_{\mathrm{p}}, \tau_{\mathrm{p}}]$. This interval captures the behavior across one full electrical period of the air-gap field.

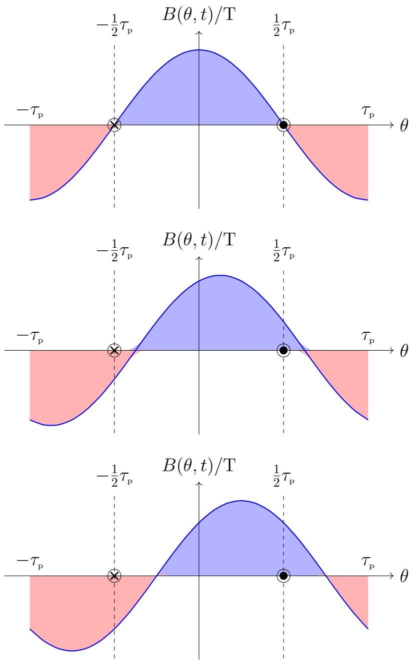

Fig. 1: Rotating field

Flux Linkage

To analyze the flux linkage with a coil, we consider a full-pitch coil. A full-pitch coil spans exactly one pole pitch. In this configuration, the coil sides are located at $x = -\frac{\tau_{\mathrm{p}}}{2}$ and $x = \frac{\tau_{\mathrm{p}}}{2}$. These positions form the integration boundaries when calculating the flux that links with the coil. The integral of the flux density $B(x,t)$ over this spatial interval defines the instantaneous flux linkage $\psi(t)$, which in turn governs the induced voltage in the coil.

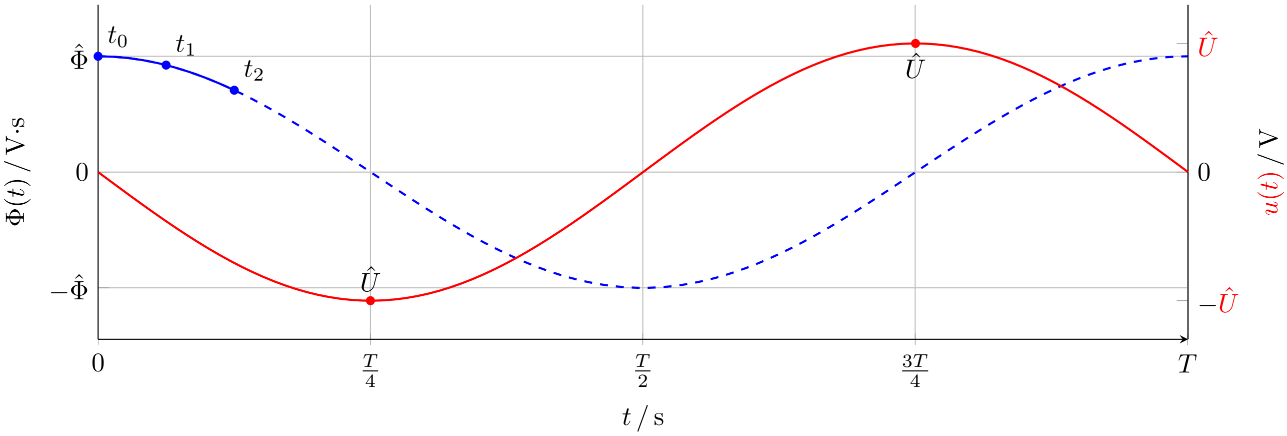

Fig. 2: Flux per pole $\Phi(t)$ over time

The maximum flux linkage per coil occurs when the zero-crossings of the flux density wave align precisely with the coil sides. In this case, the positive halve of the sinusoidal flux density is symmetrically distributed between the two sides (see figure), maximizing the net flux threading the coil area. This configuration is characteristic of a full-pitch coil, optimally placed within one pole pair.

Flux per pole

To compute the flux per pole, one must perform a volume integral of the flux density over the pole pitch and the axial length of the machine. Since B varies only along the air-gap periphery (the x-direction), and is uniform along the axial length l_{Fe}, the volume integral simplifies to a product of the axial length and a one-dimensional integral:

$$\Phi = l_{\mathrm{Fe}} \cdot \int_{-\tau_{\mathrm{p}}/2}^{\tau_{\mathrm{p}}/2} B(x) \cdot dx$$

This simplification highlights how the air-gap flux density distribution directly determines the total magnetic flux per pole.
To compute the flux per pole Φ, we integrate the flux density over the pole pitch. Assuming the air-gap flux density is sinusoidal, the following substitution simplifies the integral:

$$u = \frac{\pi x}{\tau_{\mathrm{p}}} \quad \text{and} \quad dx = \frac{\tau_{\mathrm{p}}}{\pi}\, du$$

With this substitution, the integration limits transform as follows:

$$x \in \left[-\frac{\tau_{\mathrm{p}}}{2}, \frac{\tau_{\mathrm{p}}}{2}\right] \;\rightarrow\; u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Thus, the flux per pole at $t = 0$ becomes:

$$\begin{aligned}
\Phi_0 &= l_{\mathrm{Fe}} \cdot \int_{-\tau_{\mathrm{p}}/2}^{\tau_{\mathrm{p}}/2} \hat{B} \cos\left(\frac{\pi x}{\tau_{\mathrm{p}}}\right) dx \\
&= l_{\mathrm{Fe}} \cdot \hat{B} \cdot \frac{\tau_{\mathrm{p}}}{\pi} \int_{-\pi/2}^{\pi/2} \cos(u)\, du \\
&= l_{\mathrm{Fe}} \cdot \hat{B} \cdot \frac{\tau_{\mathrm{p}}}{\pi} \cdot 2 \\
&= \frac{2}{\pi} \cdot \hat{B} \cdot \tau_{\mathrm{p}} \cdot l_{\mathrm{Fe}}
\end{aligned}$$

This expression corresponds to the **maximum flux per pole**. Recall that $0{,}636 \approx \frac{2}{\pi}$, which also represents the average area under a normalized sinusoidal curve over half a period. As time progresses (see Fig. 1), the sinusoidal flux density wave rotates, and the resulting flux linkage becomes time-dependent. The integrated flux varies as a cosine function, as illustrated in Fig. 2:

$$\Phi(t) = \hat{\Phi} \cdot \cos(\omega t), \quad \text{with } \hat{\Phi} = \Phi_0$$

To compute the induced voltage, we apply **Faraday's law**, which states that the voltage is proportional to the time derivative of the flux linkage. Differentiation of $\Phi(t)$ requires the chain rule. However, it is important to note that the maximum flux per pole, $\hat{\Phi}$, is constant in time, and therefore its derivative is zero. Applying the chain rule gives:

$$\frac{d\Phi(t)}{dt} = -\hat{\Phi} \cdot \omega \cdot \sin(\omega t)$$

Voltage Equation

Finally, accounting for the winding factor k_{w} and the number of series turns per phase N_{ph} leads directly to the familiar RMS voltage equation:

$$\begin{aligned}
U_{\mathrm{rms}} &= \omega \cdot k_{\mathrm{w}} \cdot N_{\mathrm{ph}} \cdot \frac{\hat{\Phi}}{\sqrt{2}} \\
&= \omega \cdot k_{\mathrm{w}} \cdot N_{\mathrm{ph}} \cdot \frac{2}{\pi} \cdot \frac{\hat{B}}{\sqrt{2}} \cdot \tau_{\mathrm{p}} \cdot l_{\mathrm{Fe}} \\
&= \omega \cdot \frac{\hat{\psi}}{\sqrt{2}} = \frac{\hat{U}}{\sqrt{2}}
\end{aligned}$$

Summary

This note explains how the terminal voltage in electrical machines originates from a time-varying magnetic flux linkage. Starting from the spatial distribution of the air-gap flux density, we integrate over the pole pitch to find the maximum flux per pole. As the flux density rotates, the flux linkage becomes time-dependent, following a sinusoidal pattern. Applying Faraday's law reveals that the voltage is also sinusoidal. Finally, considering the winding factor and number of turns connects the physical machine design to its electrical output.